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ON A CENTRIPETALLY LOADED MODEL SIMULATING BECK'S COLUMN

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Abstract—In this study it is shown that the non-conservative stability problem of an end loaded follower force, Beck's column, shares one frequency and an associated mode with a particular conservative centripetally loaded column. This equivalence enables one to carry out a series of experiments on the centripetally loaded columns and then infer related results for Beck's column. In this way the difficult task of experimentally producing a follower end force is avoided and the experimental verification of Beck's column is accomplished indirectly.

1. INTRODUCTION

The stability problem of non-conservative structural systems is often encountered in engineering practice, and in particular in aeronautical engineering. This problem has received considerable attention and good accounts of related analysis can be found in the literature, see, e.g. Bolotin (1961), Herrmann (1967) and Liepholz (1980). Structures subjected to follower forces form a class of typical non-conservative systems. Beck (1952) studied a cantilevered column subjected to a tangential follower force at its free end, and developed dynamic criteria for determining the critical load of the problem. This so-called Beck's column, is often used as an example to illustrate the features of non-conservative systems and newly proposed methods. There have not been satisfactory results in the literature for the experimental verification of Beck's problem, since the tangential follower force cannot be easily realized under common experimental conditions. Willems (1966) pointed out the features of a centripetally loaded column which are shared with Beck's column and he carried out the experiments on the former. The results he obtained are the static instability of the centripetally loaded model and not the flutter instability of Beck's column. This was pointed out and correct relations between the two problems were described by Hung *et al.* (1967).

In this paper, the relations of the centripetally loaded column with Beck's column are analysed in more detail, and it is shown that for a definite value of load, some centripetally loaded columns can be found which are equivalent to Beck's column with regard to one frequency and the related model. That is, the frequencies and the related modes of Beck's column can be found among a set of centripetally loaded columns. According to this analysis, a simple experimental model simulating Beck's column is established and satisfactory results are obtained.

2. DYNAMICAL CHARACTERISTICS OF BECK'S COLUMN

Beck's column is a cantilevered elastic column subjected to a tangential follower force at its free end, as shown in Fig. 1. The non-dimensional form of the boundary value problem which describes the transverse motion of the column is

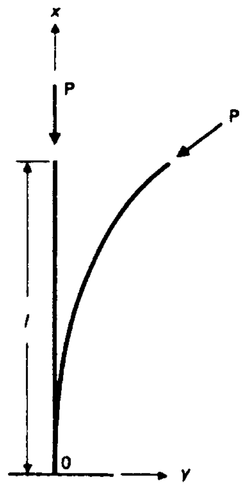

Fig. 1. Beck's column.

$$y^{(4)}(\xi)+\lambda y''(\xi)-\Omega^2 y(\xi) = 0, \qquad 0 \leqslant \xi \leqslant 1 \tag{1}$$

$$y(0) = 0 \tag{2}$$

$$y'(0) = 0 \tag{3}$$

$$y''(1) = 0 \tag{4}$$

$$y'''(1) = 0 \tag{5}$$

where λ is the non-dimensional load, and Ω^2 the non-dimensional frequency.

The general solution of eqn (1) is

$$y(\xi) = A \cosh \beta_1\xi + B \sinh \beta_1\xi + C \cos \beta_2\xi + D \sin \beta_2\xi \tag{6}$$

where

$$\beta_1 = [(\Omega^2+\lambda^2/4)^{1/2}-\lambda/2]^{1/2}, \qquad \beta_2 = [(\Omega^2+\lambda^2/4)^{1/2}+\lambda/2]^{1/2}. \tag{7}$$

Substituting eqn (6) into homogeneous boundary conditions (2)–(5) gives the characteristic equation of Beck's column

$$J_B(\lambda, \Omega^2) = 2\Omega^2(1+\cosh \beta_1 \cos \beta_2)+\lambda\Omega \sinh \beta_1 \sin \beta_2+\lambda^2 = 0. \tag{8}$$

The eigenfrequencies of the column with different values of load can be determined from eqn (8). However, we are more concerned with the vibrating modes of Beck's column.

From eqns (2)–(4), we have the mode

$$y(\xi) = A[\cosh \beta_1\xi - M \sinh \beta_1\xi - \cos \beta_2\xi + N \sin \beta_2\xi] \tag{9}$$

where A is an undetermined coefficient, and

$$M = (\beta_1^2 \cosh \beta_1+\beta_2^2 \cos \beta_2)/(\beta_1^2 \sinh \beta_1+\beta_1\beta_2 \sin \beta_2), \qquad N = M\beta_1/\beta_2. \tag{10}$$

Also, we have

$$y'(\xi) = A[\beta_1 \sinh \beta_1\xi - M\beta_1 \cosh \beta_1\xi + \beta_2 \sin \beta_2\xi + N\beta_2 \cos \beta_2\xi]. \tag{11}$$

The actual motion of the column is

$$Y(\xi, t) = y(\xi)\, e^{i\omega t}. \tag{12}$$

It is noted that the ratio of $Y(\xi, t)$ over $Y'(\xi, t)$ is independent of time t, i.e.

$$\frac{Y(\xi,t)}{Y'(\xi,t)} = \frac{y(\xi)}{y'(\xi)} = \frac{\cosh \beta_1\xi - M \sinh \beta_1\xi - \cos \beta_2\xi + N \sin \beta_2\xi}{\beta_1 \sinh \beta_1\xi - M\beta_1 \cosh \beta_1\xi + \beta_2 \sin \beta_2\xi + N\beta_2 \cos \beta_2\xi} = R(\xi). \tag{13}$$

This indicates that at any time t, during the motion, the tangential line of any point ξ on the column always passes through a fixed point on the undeformed axis, the distance from which to the point ξ is $R(\xi)$. In particular, at the free end of the column, $\xi = 1$, we have

$$y(1)/y'(1) = R(1) = r/l. \tag{14}$$

Because $y(\xi)$ is one of the vibrating modes of Beck's column and r/l is a constant, eqn (14) shows that for any one of the modes, the tangential follower force of Beck's column is equivalent to a centripetal force acting on a cantilevered column, as shown in Fig. 2, the line of action of which always passes through a fixed point on the undeformed axis and along the tangential direction of the deformed axis at the free end. The variable r is the distance from the fixed point to the free end.

3. CENTRIPETALLY LOADED COLUMN AND ITS RELATION TO BECK'S COLUMN

In this section, we shall analyse a general centripetally loaded model, as shown in Fig. 3. The non-dimensional form of its dynamical equation and boundary conditions are

$$\begin{aligned} &y^{(4)}(\xi) + \lambda y''(\xi) - \Omega^2 y(\xi) = 0, \qquad 0 \leqslant \xi \leqslant 1 \\ &y(0) = y'(0) = y''(1) = 0 \\ &y'''(1) + \lambda[y'(1) - y(1)/(r/l)] = 0. \end{aligned} \tag{15}$$

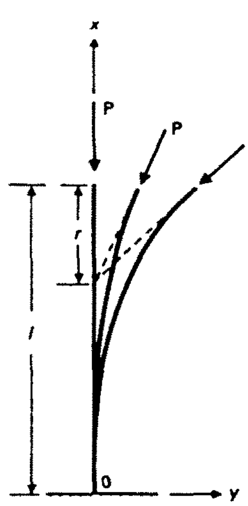

Fig. 2. A vibrating mode of Beck's column.

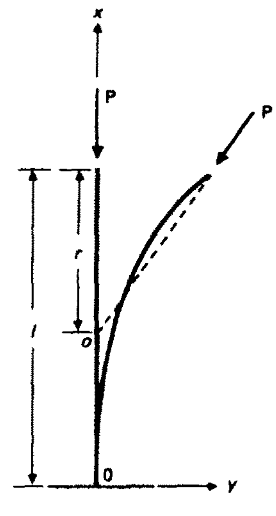

Fig. 3. A centripetally loaded model.

Using a similar procedure as in the preceding section, we obtain the characteristic equation of this model as

$$J_C(\lambda,\Omega^2,r/l) = (2\Omega^2+\lambda^2)(1+\cosh\beta_1\cos\beta_2)-\lambda^2-\lambda\Omega\sinh\beta_1\sin\beta_2$$
$$-\lambda(\beta_1^2+\beta_2^2)\left(\frac{1}{\beta_1}\sinh\beta_1\cos\beta_2-\frac{1}{\beta_2}\cosh\beta_1\sin\beta_2\right)\bigg/(r/l)=0 \quad (16)$$

where r/l is a variable parameter.

If the value of r/l satisfies the condition in eqn (14), then eqn (16) degenerates into the characteristic equation of Beck's column given in eqn (8). That is, the centripetally loaded column is equivalent to Beck's column under the condition of eqn (14). However the equivalence holds true with respect to a single frequency and the related mode.

With the centripetally loaded column, one may obtain any one of the frequencies and the corresponding modes of Beck's column, from which the critical values may be determined. The key problem remains as to how to determine the appropriate values of r/l under which the two problems are equivalent. An iterative method for calculating the values of r/l and thus the desired frequencies has been proposed (Xiang and Wang, 1988a,b). The convergence of the iteration is quite rapid.

Table 1 shows the characteristic values of the centripetally loaded column with different values of r/l, in which the values underlined are completely the same as those of Beck's column.

Table 1. Characteristic values of a centripetally loaded column with different values of r/l

	λ/π^2				
r/l $\quad \Omega^2/\pi^4 =$	0.0	0.5	1.0	1.5	2.0
0.22964	4.98442	4.29651	3.87921	3.90281	4.24808
0.25968	4.98442	4.17695	3.56673	3.37517	3.58243
0.29767	4.98442	4.04402	3.21719	2.74990	2.78699
0.39356	4.98442	3.84944	2.71722	1.75463	1.54161
0.46652	4.98442	3.75969	2.50109	1.27311	0.99169
0.58816	0.12690	0.23516	0.34127	0.44149	
0.64950	0.12690	0.20205	0.27195	0.29709	
0.69308	0.12690	0.18174	0.22758	0.19084	

Table 2. Geometric size of specimens (mm)

No.	l	b	h
1	299.94	10.21	2.85
2	300.10	10.11	2.92
3	300.34	10.29	2.87

4. EXPERIMENTAL VERIFICATION

4.1. *Experimental model*

Since it is much easier to realize a centripetal load in experiments than a tangential follower force, and the centripetally loaded column has relations with Beck's column, one may obtain the experimental results for the frequencies of the latter by carrying out experiments on the former. However, the critical load of Beck's column cannot be obtained directly through such experiments, because it is impossible for a centripetally loaded column to lose stability by flutter.

In our experiments we measured a few frequencies of the centripetally loaded column with appropriate values of the load P and the distance r/l, which are the same as those of Beck's column according to the analysis outlined earlier. Using the finite number of experimental results obtained we employed a curve fitting procedure. From the fitted curve the onset of loss of stability for Beck's column can be estimated.

Suppose now that the characteristic equation of Beck's column takes the following form:

$$J_s(\lambda, \Omega^2, a_i) = 2a_1\Omega^2(a_2 + a_3 \cosh a_4\beta_1 \cos a_5\beta_2) + (a_6\lambda)(a_7\Omega) \sinh (a_8\beta_1) \sin (a_9\beta_2) + a_{10}\lambda^2 = 0 \quad (17)$$

where a_i are some undetermined constants. It is easy to see that when all of the values of a_i are unity, eqn (17) will be the same as the exact characteristic eqn (8).

Using the least squares curve-fitting method for the experimental data obtained, we may determine the values for a_i. Following this we may calculate the minimum value of load through eqn (17) under which the first two frequencies are coincident, and this value of load may be taken as the experimental critical load for Beck's column.

It should be pointed out that Willems (1966) carried out the experiments using similar models. Since he was concerned with the critical load alone, and the two problems under consideration have essentially different modes of instability, his assertion is not correct (Huang *et al.*, 1967).

4.2. *Experiments and results*

Three aluminum specimens were made which have the elastic modular $E = 6.98 \times 10^{10}$ N m^{-2}, mass density $\rho = 2.8 \times 10^3$ kg m^{-3}, and geometric size as shown in Table 2.

The centripetal load was applied by means of thin steel wires passing through a fixed point 0′ at a distance r from the free end, which is determined from eqn (14). Figures 4 and

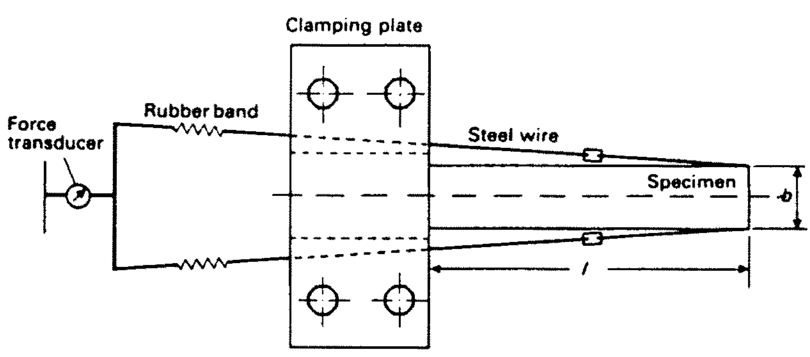

Fig. 4. Test layout—vertical view.

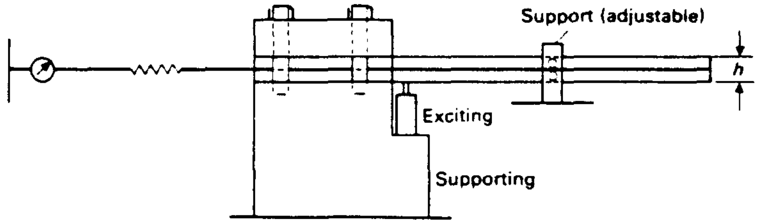

Fig. 5. Test layout—horizontal view.

5 show the test layout. At each load application the column was given a slight vibration and the resulting frequency was measured.

In order to keep the magnitude of the applied load constant during the vibration, the steel wires were connected to some rubber bands. The bands were tensioned such that they could be stretched with a very small increase in tension. Thus for all practical purposes the load on the column can be considered to be constant.

For each of the three specimens, the frequencies measured under the loads, lower than the critical value, are shown in Table 3. The non-dimensional characteristic values were then calculated and the fitting parameters a_i in eqn (17) were determined using the least squares curve-fitting method. These results are shown in Fig. 6.

Using the characteristic equation, eqn (17), with parameters a_i determined, the experimental critical load for each specimen was calculated. The non-dimensional values of these loads were found to be 1.9154 for specimen number 1, 1.8854 for specimen number 2, and 1.8720 for specimen number 3. They represent, respectively, approximately 94, 93 and 92% of the theoretical value, namely 2.0316.

Table 3. Experimental results: ω_T, theoretical; ω_E, experimental

No.	P (kg)	First frequency (Hz) r (mm)	ω_T	ω_E	Second frequency (Hz) r (mm)	ω_T	ω_E
	0.0		25.55	25.33		160.08	154.27
	1.538	216.06	26.44	26.28	63.840	157.90	151.36
	3.077	214.16	27.39	27.14	65.000	155.66	149.10
	6.153	210.08	29.45	29.50	67.510	151.03	145.78
1	9.230	205.55	31.76	31.80	70.310	146.16	151.74
	12.306	200.49	34.38	34.30	73.460	140.98	137.02
	15.383	194.81	37.39	37.42	77.070	135.42	133.89
	18.459	188.32	40.91	41.01	81.270	129.37	129.70
	21.536	180.76	45.14		86.310	122.63	121.10
	0.0		26.15	25.50		163.84	155.48
	1.637	216.22	27.06	26.71	63.895	161.61	152.44
	3.273	214.33	28.03	28.25	65.055	159.32	150.97
	6.546	210.24	30.14	29.82	67.565	154.58	146.64
2	9.819	205.70	32.50	32.85	70.525	149.55	142.02
	13.092	200.81	35.18	35.76	73.685	144.29	138.60
	16.365	195.12	38.27	38.37	77.295	138.60	133.84
	19.638	188.62	41.87	42.04	81.545	132.41	129.22
	22.911	181.06	46.20		86.385	125.51	124.27
	0.0		25.60	25.19		160.78	153.91
	1.579	216.34	26.56	25.66	63.913	158.58	151.41
	3.158	214.44	27.51	26.81	65.083	156.34	149.05
	6.316	210.35	29.57	28.92	67.593	151.69	145.17
3	9.474	205.81	31.89	30.97	70.393	146.79	141.23
	12.632	200.75	34.53	34.76	73.553	141.59	136.99
	15.790	195.06	37.55	37.88	77.163	136.01	133.15
	18.947	188.56	41.09	41.42	81.373	129.93	127.71
	22.105	181.00	45.34		86.423	123.16	123.38

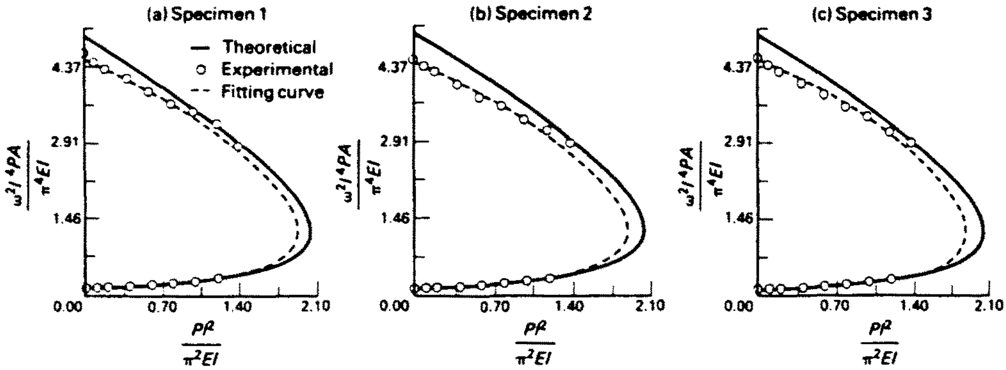

Fig. 6. Theoretical, experimental and fitting results for Becks's column.

5. CONCLUDING REMARKS

By analysing the dynamic characteristics of Beck's column and a centripetally loaded model, it has been shown that the two problems have equivalence with regard to any one single frequency and the related mode. It is possible to use this fact to obtain the frequencies of Beck's column and its critical load.

According to this analysis, a simple experimental model simulating Beck's column has been established, and in effect a non-conservative problem has been transformed into a set of conservative problems for experimental verification. Satisfactory results have been obtained and verified.

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